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Reply

Accurate and Efficient Explicit Approximations of the Colebrook Flow Friction Equation Based on the Wright ω-Function: Reply to Discussion

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Abstract: This reply gives two corrections of typographical errors in respect to the commented article, and then provides few comments in respect to the discussion and one improved version of the approximation of the Colebrook equation for flow friction, based on the Wright ω-function. Finally, this reply gives an exact explicit version of the Colebrook equation expressed through the Wright ω-function, which does not introduce any additional errors in respect to the original equation. All mentioned approximations are computationally efficient and also very accurate. Results are verified using more than 2 million of Quasi Monte-Carlo samples.

Keywords: Colebrook equation; hydraulic resistance; Lambert W-function; Wright ω-function; explicit approximations; computational burden; turbulent flow; friction factor

1. Introduction

This reply provides responses to the discussion related to the commented article [1].

Recently we published a paper [1], for which we received an interesting discussion [2] and we would like to thank the authors for the useful comments. Our reply will offer two corrections of typographical errors in respect to our commented article [1], then provide a few comments in respect to the discussion [2]. Finally, one improved version of approximations of the Colebrook equation for flow friction will be presented, which is also based on the Wright ω-function. Results in this reply are verified using more than 2 million of Quasi Monte-Carlo samples [3].

2. Typographical Corrections

In our recent paper [1] we found that one pair of parentheses is missing in Equation (2). The corrected Equation (2) of the commented article [1] is here given as Equation (1):

$$\left.\begin{array}{r}\frac{1}{\sqrt{f}}=\frac{2}{ln(10)}\cdot\left(ln\left(\frac{R}{2.51}\cdot\frac{ln(10)}{2}\right)+W(e^{x})-x\right)\\ x=ln\left(\frac{R}{2.51}\cdot\frac{ln(10)}{2}\right)+\frac{R\cdot\varepsilon^{*}}{2.51\cdot 3.71}\cdot\frac{ln(10)}{2}\end{array}\right\} \quad (1)$$

As reported in the discussion [2], in Equation (11) of the commented paper [1], parentheses are also missing, while the correct version is given by Equation (A7), in Appendix A of the commented paper [1]. The corrected Equation (11) of the commented article [1] is here given as Equation (2):

$$B\approx s\cdot\left(0.0001086\cdot s^{6}+0.9824\right)-\frac{0.006206}{r}-r\cdot(0.000007237\cdot r-0.006656)+1.881 \quad (2)$$

The nature of these errors is typographical and does not affect other formulas or findings. We apologize to the readers for any inconvenience caused by these two typographical errors, and we want to thank the authors of the discussion [2] for pointing out the errors in the second here corrected equation.

3. Observations Related to the Discussion [2]

Here we would like to underline the following:

-To avoid misunderstanding, ε^* in the discussed paper [1] represents the dimensionless relative roughness of inner pipe surface, while R represents the dimensionless Reynolds number. The discussion [2] uses ε^* but also $\frac{\varepsilon}{D}$ for the dimensionless relative roughness of the inner pipe surface and R but also Re for the dimensionless Reynolds number.

-The discussion [2] is flawed with typographical errors such as: In the title of the discussion [2], "DejanBrkić" should be "Dejan Brkić", in Equation (6) of [2] lower-case ***c*** should be capital-case ***C***, also in Equation (10) of [2] "1.0119.C" should be "1.0119·C", "Right" should be "Wright", etc.

-The index "CW" in f_{CW} and the term "Colebrook-White equation" in the discussion [2] is used in the same meaning as f and the term "Colebrook" in [1]. The source for of Equation (1) of the discussion [2] is not Colebrook and White [4], but Colebrook [5] (the Colebrook equation [5] is based on the experiment of Colebrook and White [4]).

-We believe the term "deviation" used in the discussion [2], does not mean the square root of the variance as it is common in statistics, but it is the relative error as defined in [1] (also by Equation (5) of the discussion [2]).

-Eqs. (2), (3) and (4) with the reference to the discussion [2] and Eqs. (3), (5) and (6) with the reference to [1] gives the relative error of no more than 0.152%, 0.0552%, and 0.0096%, respectively. Here we use 2^{21}~2.1 million of Quasi Monte-Carlo samples for this verification using the LPTAU51 algorithm [3]. Previously in [1], using the methodology from [6], only 740 points in Microsoft Excel were used; this error was estimated to be 0.13%, 0.045%, and 0.0096%, respectively.

-Equation (A7) of [1] introduces a Padé-based approximation, which replaces the original term $B \approx ln\left(\frac{R}{2.18}\right) \approx ln(R) - 0.7794$ by simple polynomials. The additional relative error of this approximation is 0.08%. Recall that Equation (A7) of [1] is also Equation (2) of this reply. The approximation uses polynomial expressions instead of the computationally expensive logarithmic function [7]. The relative error of the approximations given by Equations (2)–(4) with the reference to the discussion [2] and Equations (3), (5) and (6) with the reference to [1] is no more than 0.403%. We used 2^{21}~2.1 million of Quasi Monte-Carlo samples for this verification [3].

-We are concerned that disputed accuracy of some of our findings is based on the paper [8], which is written by the same authors as the discussion paper [2]. In [8], authors claim that explicit approximations of the Colebrook equation given by Serghides [9] and Buzzelli [10] are very inaccurate. These findings in [8] are in contradiction with [6,11–16] where they are classified as very accurate.

-We believe that to date, the most accurate explicit approximation in respect to the Colebrook equation is by Vatankhah [17] with the relative error of no more than 0.0028%, which is more accurate in comparison to Equation (10) of the discussion [2]. Using our methodology for the estimation of the relative error [1], which uses the same methodology as in [6], the relative error of Equation (10) of the discussion [2] is estimated up to 0.1928%. Our findings are tested using 2048 quasi-random points [17] which are not sufficient according to [2]. However, it is true that using 2^{21}~2.1 million of Quasi Monte-Carlo samples for verification, some points with slightly elevated error can be found.

-Term "trial-error methods" used in the discussion [2] is more likely "iterative methods" in [1] as explained in [18–20].

-Figures 1, 3, and 5 of the discussion [2] shows the constant value of the relative error over the domain of the relative roughness of the inner pipe surface, which is not realistic and they seem to be in contradiction with [9–17].

4. Colebrook Equation Expressed through the Wright ω-Function

Equation (1) of this reply, i.e., Equation (2) of [1], can be expressed through the Wright ω-function as Equation (3):

$$\left.\begin{array}{c} \frac{1}{\sqrt{f}} = \frac{2}{ln(10)} \cdot \left(ln\left(\frac{R}{2.51} \cdot \frac{ln(10)}{2} \right) + \omega(x) - x \right) \\ x = ln\left(\frac{R}{2.51} \cdot \frac{ln(10)}{2} \right) + \frac{R \cdot \varepsilon^*}{2.51 \cdot 3.71} \cdot \frac{ln(10)}{2} \end{array}\right\} \quad (3)$$

Both equations, Equation (1) of this reply, i.e., Equation (2) of [1], which express the Colebrook equation through the Lambert W-function, and Equation (3), which gives the Colebrook equation through the Wright ω-function, do not introduce any additional errors compared to the original equation given by Colebrook [5]; Equation (1) of [1]. However, the Lambert W-function and the Wright ω-function can be evaluated only approximately. We use the Wright ω-function implemented in Matlab as WrightOmegaq [21] for verifications of all approximations in this reply. However, to date, the Lambert W-function has been more commonly used in hydraulics [22–26] compared to its cognate Wright ω-function. Unfortunately, built-in procedures for these special functions do not exist in common spreadsheet solvers, such as Microsoft Excel [27]. However, Microsoft Excel functions [28] of high accuracy can be made for these special mathematical functions using Visual Basic for Applications (VBA) programming environment. Consequently, Equation (1) of this reply, i.e., of Equation (2) of [1], or Equation (3) from this section can be used for real calculations. Likewise, the approximations of $y = W(e^x) - x = \omega(x) - x$ from Table 1 of this reply or from [1] can be used for real calculations, also [29].

Table 1. A series expansion y of the $W(e^x) - x$ and related approximation of the Colebrook equation based on it.

y=$W(e^x)$−x	Approximation	δ%	Eq.
$y \approx -ln(x) + \frac{ln(x)}{x} + 0.000818$	$\frac{1}{\sqrt{f}} \approx 0.8686 \cdot \left[B - C + \frac{C}{B+A} + 0.000818 \right]$	0.136%	(4)

$A \approx \frac{R \cdot \varepsilon^*}{8.0878}$, $B \approx ln\left(\frac{R}{2.18}\right) \approx ln(R) - 0.7794$, $C = ln(B + A)$, and $\delta\ \% = (\,|f_{accurate} - f\,|/f_{accurate}) \cdot 100\%$; R is the Reynolds number while ε^* is the relative roughness of the inner pipe surface, both dimensionless, while $\delta\%$ denotes the percentage relative error (here evaluated in Matlab using more than 2 million of quasi Monte-Carlo samples)

5. A New Updated Explicit Approximation

To construct an explicit approximation of the Colebrook equation using the procedure from [1], the term $y = W(e^x) - x$ of Equation (1), i.e., of the corrected Equation (2) of [1], based on [26] should be very accurately approximated using the series expansion of y. Table 1 gives an approximation of $W(e^x) - x$; Equation (4) with its estimated relative error. We analyzed approximations of the form $y \approx -ln(x) + \frac{ln(x)}{x} + d$, where d is a real constant obtained in order to minimize the maximal relative error of the Colebrook equation. The original approximation given by Equation (3) of [1] can be viewed as a variant with $d = 0$. Results of the Microsoft Excel solver and the subsequent verification in Matlab using more than 2 million of quasi Monte-Carlo samples indicate that $d = 0.000818$ in Equation (4) decreased the relative error from 0.152% to 0.136%.

The distribution of the relative error by the proposed explicit approximation of Colebrook's equation; Equation (4) from Table 1 is given in Figure 1. Note that Figure 1 is made in Microsoft Excel using 740 points [6], and therefore $\delta\%_{max}$ is estimated not to be more than 0.098641% while in the 2 million of quasi Monte-Carlo sample [3] a pick of error $\delta_{max} = 0.136\%$ is detected for $R = 4000$ and $\varepsilon^* = 4.6 \cdot 10^{-7}$.

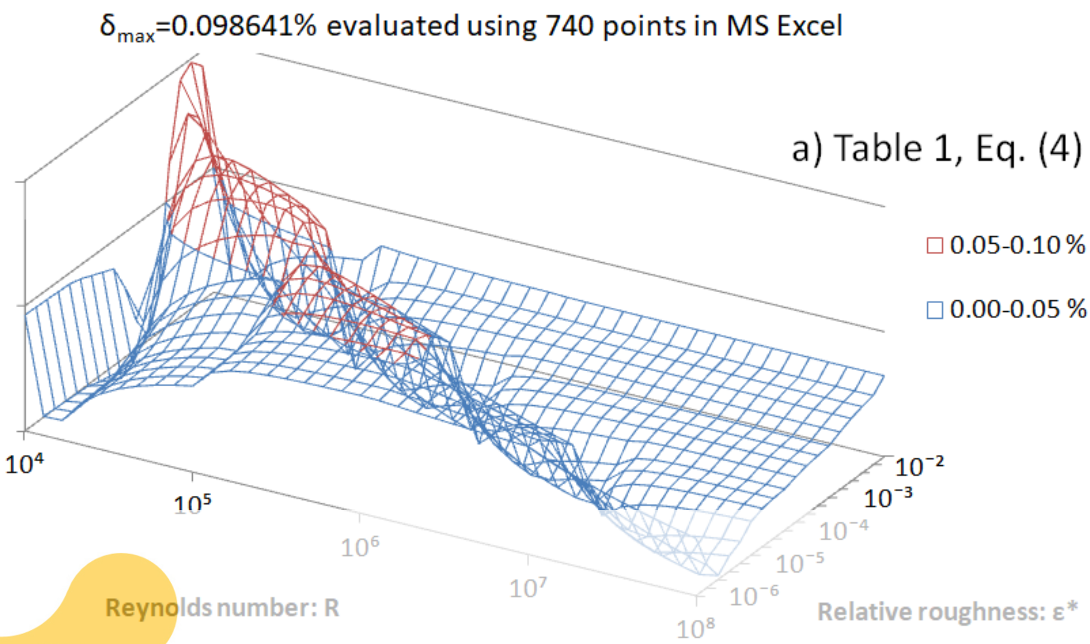

Figure 1. Distribution of the relative error by the proposed explicit approximation of the Colebrook's equation; Equation (4) from Table 1 using a limited number of 740 sample points.

6. Conclusions

In addition to our paper [1] and using the same procedures, here we verified our explicit approximations of the Colebrook equation for flow friction using more than 2 million of Quasi Monte-Carlo samples. These approximations are also computationally simple, as they contain only one or two computationally expensive logarithms [1,7,14,15,26]. Although much more samples than in [1] were used for verifications, as proposed in [2], the results presented here are consistent with the original paper [1].

To avoid misinterpretations, here we present the Colebrook equation expressed directly through the Wright ω-function; Equation (3) (while in [1] it was through the Lambert W-function [30]; Equation (2) of [1]; here Equation (1)).

Presented approximations can be used for flow friction simulators to speed up calculations [26], and also for faster modelling of pipe and conduit networks [31–35].

Author Contributions: Based on the expansion of functions provided by P.P., D.B. developed approximations. Regression analyses in Eureqa and verifications in Matlab [computer software] are performed by P.P. D.B. wrote a draft of this reply.

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Abbreviations

The following symbols are used in this reply:

Variables	
A	variable that depends on R and ε^* (dimensionless)
B	variable that depends on R (dimensionless)
C	variable that depends on variables A and B (dimensionless)
d	real constant (dimensionless)
f	Darcy (Moody) flow friction factor (dimensionless)
R	Reynolds number (dimensionless)
r	variable that depends on R (dimensionless)
x	variable in function on R and ε^* (dimensionless)
y	variable in function on x (dimensionless)
ε^*	relative roughness of inner pipe surface (dimensionless)
δ	relative error (%)
Functions	
ln	natural logarithm
s	Padé approximant
W	Lambert W-function
ω	Wright ω-function

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